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Draw a cross through the box (☒) if you have NOT written in this booklet

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OUTSTANDING SCHOLARSHIP

NZQA

Mana Tohu Mātauranga o Aotearoa
New Zealand Qualifications Authority

Scholarship 2025
Calculus

Time allowed: Three hours
Total score: 32

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

You should answer ALL the questions in this booklet.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Correct answers only will not be sufficient

If you need more room for any answer, use the extra space provided at the back of this booklet.

Check that this booklet has pages 2–24 in the correct order and that none of these pages is blank.

Do not write in any cross-hatched area (). This area will be cut off when the booklet is marked.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Find all the real solutions of the following equation:

$$\sqrt[3]{125^{x^3-5x^2+11x-3}} = \sqrt[4]{5^{4x^2+12}}$$

$$\log_5\left(\sqrt[3]{125^{x^3-5x^2+11x-3}}\right) = \log_5\left(\sqrt[4]{5^{4x^2+12}}\right)$$

$$\Rightarrow \frac{1}{3}\log_5\left(125^{x^3-5x^2+11x-3}\right) = \frac{1}{4}\log_5 5^{4x^2+12}$$

$$\Rightarrow x^3-5x^2+11x-3 = x^2+3$$

$$\Rightarrow x^3-6x^2+11x-6=0$$

$x=1$ is a solution

$$(x-1)(x^2-5x+6)=0$$

$$(x-1)(x-2)(x-3)=0$$

$$x \in \{1,2,3\}$$

(b) Find all the complex solutions of the following equation:

$$z = 2 - \frac{2+\mathrm{i}}{\mathrm{i}z}$$

$$z^2 = 2z - (2+i)(-i)$$

$$z^2 = 2z - (-2i+1)$$

$$= 2z + 2i - 1$$

$$z^2 - 2z - 2i + 1 = 0$$

$$(z-1)^2 - 1 - 2i + 1 = 0$$

$$(z-1)^2 = 2i$$

$$z-1 = \pm(i+1)$$

$$z = 1 \pm (i+1)$$

$$z \in \{2+i, -i\}$$

(c) How many different complex solutions does the equation $z^{2025} = \overline{z}$ have?

Justify your answer.

Note: as real numbers and purely imaginary numbers are both subsets of the complex numbers, you should consider them also.

$|\bar{z}| = |z|$

but since $z^{2025} = \bar{z}$

$|z^{2025}| = |z|^{2025} = |\bar{z}| = |z|$

so $|z|^{2025} = |z|$

$|z| = 1$ or $|z| = 0$

if $|z| = 0$, $z = 0$

else

$|z| = 1$, $\bar{z} = \frac{1}{z}$, $z^{2025} = \frac{1}{z}$, $z^{2026} = 1$

$(e^{i\theta})^{2026} = e^{i(2n\pi)}$

$e^{2026 i\theta} = e^{i2n\pi}$

$\theta \in \{\frac{\pi}{1013}, \frac{2\pi}{1013}, \ldots \ldots \frac{2026\pi}{1013}\}$

~~lot~~ 2026 ~~roo~~ roots of unity

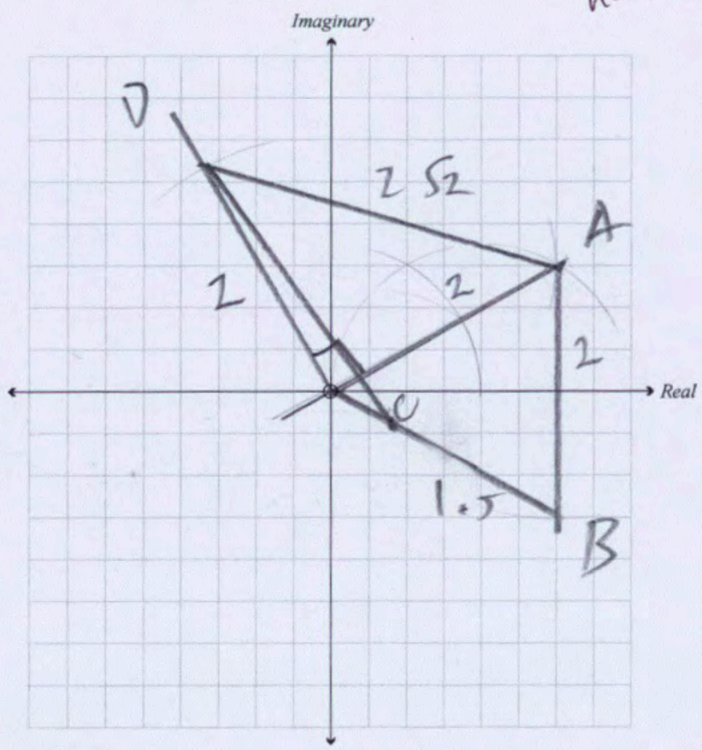
so 2027 solutions total.

(d) Let $z = r\,\text{cis}\,\theta$ be a complex number with $r > 1$ and $0 < \theta < \frac{\pi}{2}$.
Let the point A represent z in the Argand diagram and the points B, C, and D represent the complex numbers $\overline{z}$, $\frac{1}{z}$, and $\mathrm{i}z$ respectively.

If $\angle ABC = 60°$ and BC = 1.5, find the area of the quadrilateral ABCD.
Use the Argand diagram below to support your working.

If only Nico Mckinlay were here for this

H T ~~Mckay~~ when he gets to do complex bash in school call



AB is parallel to Imaxis since B is a reflection of A about Real axis

For the same reflection, $\angle ABO = \angle BAO$

C lies on ~~the same li~~ $\overline{OB}$ as $\bar{z} = \frac{r^2}{z}$
and Between OB as $r > 1$
so $\angle ABC = \angle ABO$
thus $\angle ABO = \angle BAO = 60°$
At $\angle AOB = 60°$

so ABO equilateral and $\theta = 30°$

Since BC = 1.5 and B and C share
on argand,

$BC = |z| - |\frac{1}{z}| = r - \frac{1}{r} = 1.5$

$r^2 - 1.5r - 1 = 0$

$r = 2$ or $-\frac{1}{2}$ ← disregard

$r = 2$

so $z = \cancel{\ }2e^{i\frac{\pi}{6}}$

AOD is a right angle isosceles triangle
so $AD = 2\sqrt{2}$

umm can i use the shoe lace formula
$z = \sqrt{3} + i$, $\bar{z} = \sqrt{3} - i$, $z^{-1} = \frac{\sqrt{3}}{4} - \frac{i}{4}$, $iz = -1 + \sqrt{3}i$

$A = (\sqrt{3}, 1)$, $B = (\sqrt{3}, -1)$ $C = (\frac{\sqrt{3}}{4}, -\frac{1}{4})$ $D = (-1, \sqrt{3})$

$2 \times \text{Area} = |x_1 y_2 + x_2 y_3 + x_3 y_4 + x_4 y_1 - x_2 y_1 - x_3 y_2 - x_4 y_3 - x_1 y_4|$

$= |-\sqrt{3} - \frac{\sqrt{3}}{4} + \frac{3}{4} - 1 - \cancel{4}\sqrt{3} + \frac{\sqrt{3}}{4} - \frac{1}{4} - 3|$

$= \frac{7 + 4\sqrt{3}}{2}$

So Area $= \frac{7 + 4\sqrt{3}}{4} = \frac{7}{4} + \sqrt{3}$

See back page

QUESTION TWO

(a) A function is defined parametrically by the equations:

$x = 2t + t^2$

$y = p(t)$

where:

- $p(t) = at^3 + bt^2 + ct + d$
- $p(1) = 2.5$
- $p'(1) = 6$

Given that $\dfrac{d^2y}{dx^2} = \dfrac{3}{4(1+t)}$, find the value of $p(2)$.

$\dfrac{d^2y}{dx^2} = \dfrac{d\left(\frac{dy}{dx}\right)}{dt} \times \dfrac{dt}{dx}$ So ~~$\frac{dx}{dt}$~~

$\dfrac{dx}{dt} = 2+2t$

so $\dfrac{3}{4(1+t)}(2+2t) = \dfrac{d\left(\frac{dy}{dx}\right)}{dt}$

$= \dfrac{3}{2}$

which means $\dfrac{dy}{dx} = \dfrac{3}{2}t + f$

where f is a constant

$\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = \left(\dfrac{3}{2}t + f\right)(2+2t)$

$= 3t^2 + 3t + 2ft + 2f$

Since $p'(1) = 6$, $3+3+2f+2f = 6$

$f = 0$

So $\dfrac{dy}{dt} = 3t^2 + 3t$

so $y = t^3 + \frac{3}{2}t^2 + d$

Since $p(1) = \frac{5}{2}$

$1 + \frac{3}{2} + d = \frac{5}{2}$, $d = 0$

so $p(2) = 8 + \frac{3}{2} \times 4 = 14$

$y = t^3 + \frac{3}{2}t^2$

$\dot{y} = 3t^2 + 3t$

$\dot{x} = 2 + 2t$

$\frac{dy}{dx} = \frac{3t^2+3t}{2+2t} = \frac{3}{2}t$

$\frac{d\left(\frac{dy}{dx}\right)}{dt} = \frac{3}{2}$, gradient works

(b) A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in its domain.

Examples of odd functions include $f(x) = x^3$ and $f(x) = \sin(x)$.

Consider an **odd** function f that satisfies the condition $f(1-x) = f(1+x)$ for all real numbers.

Given that $f(1) = 2025$, find the value of $f(1) + f(2) + \ldots + f(2025)$.

$f(-1) = -2025$

let x be 2

$f(3) = f(-1) = -2025$

the $f(-3) = 2025$

let $x = 4$

$f(\cancel{4}5) = f(-3) = 2025$

~~Since~~

so i reckon $f(1)$, $f(5)$, $f(9)$ etc is 2025

and 3, 7, 11 etc are -2025

$f(0) = -f(-0)$

so $f(0) = -f(0) = 0$

let $x = 1$, $f(2) = f(0) = 0$

so $f(-2) = 0$

so let $x = 3$, $f(4) = 0$

Proof by induction that $f(2k) = 0$

for all $k \in \mathbb{Z}^+$

~~Sum~~ base case done

so $f(2k) = 0 \Rightarrow f(-2k) = 0$

let $x = 2k+1$, $f(2k+2) = f(-2k) = 0$

done because $f(2(k+1)) = 0$

proof by induction that $f(4k+1)$ is always 2025 and $f(4k-1)$ is always -2025 for $k \in \mathbb{Z}^+$

base case done

let $f(4k+1) = 2025$

$f(-4k-1) = -2025$

let $x = 4k+2$

$f(4k+3) = -2025 = f(4(k+1)-1)$

so $f(-4k-3) = 2025$

let $x = 4k+4$

$f(4k+\cancel{4}5) = f(4(k+1)+1) = 2025$

done

so $f(2k) = 0$ so $f(2)+f(4)+f(6)+\ldots+f(2024) = 0$

$f(1)$ cancels with $f(3)$ etc

(because $f(1)+f(3)=0$, $f(5)+f(7)=0$ $\ldots$ $f(2021)+f(2023)=0$

$f(2025)$ remains

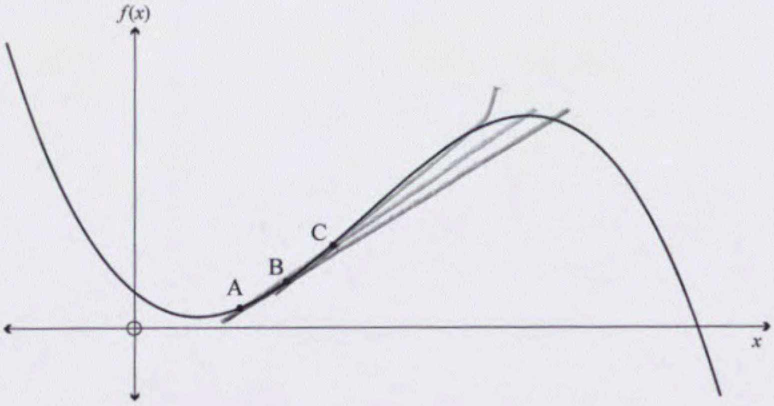
so Sum $= 2025$ because $f(2025)=2025$

FF's in school calc? Did ~~they~~ the NZMOC write this one?

this is so sigma ohio aura skibidi

(c) Consider the cubic function $f(x) = ax^3 + bx^2 + cx + d$.

The points A(2,4), B(3,9), and C(4,16) all lie on the graph of the cubic, as shown.



Suppose that the lines AB, AC, and BC intersect the cubic again at points D, E, and F, respectively, and that the sum of the x-coordinates of D, E, and F is 25.

Find the y-intercept of the cubic.

Hint: consider the function $g(x) = f(x) - x^2$ in factorised form.

~~[illegible]~~

def $g(x) = f(x) - x^2$

~~ge~~ $f(2) = 4$

$f(3) = 9$

$f(4) = 16$

so $g(2) = g(3) = g(4) = 0$

since cubic / only 3 factors

so $g(x) = a(x-2)(x-3)(x-4)$

$g(x) = a(x^3 - 9x^2 + 26x - 24)$

$f(x) = a(x^3 - 9x^2 + 26x - 24) + x^2$

eqn for AB is $y = 5x - 6$

BC is $y = 7x - 12$

AC is $y = 6x - 8$

$f(x_D) = 5x_D - 6$
$f(x_E) = \cancel{8}\,6x_E - 8$
$f(x_F) = 7x_F - 12$

well, $f(x) - 5x + 6$ has sols at A, B and D
let x coordinates of ABCDEF be $\alpha, \beta, \gamma, \delta, \varepsilon, \varphi$
Could i use vieta?
let the roots of $f(x) - 5x + 6 = \alpha, \beta, \cancel{\gamma}\ \delta$
let α be 2, β be 3 $\quad \delta = 4 - \frac{1}{a}$

$\alpha + \beta + \delta = 9 - \frac{1}{a}$ so δ is at $4 - \frac{1}{a}$
(since x^2 coefficient is $-9a+1$)

but all of $f(x) - 5x + 6$
$f(x) - 6x + 8$
$f(x) - 7x + 12$ $(-9a+1)$
have the same x^2 term
so vieta works the same on them,
so E has x coordinate $3 - \frac{1}{a}$
F has x coordinate $2 - \frac{1}{a}$
$\varepsilon = 3 - \frac{1}{a}$, $\varphi = 2 - \frac{1}{a}$
so sum is $\cancel{\frac{27}{a} - 18 = 25}$

$\cancel{\frac{27}{a} = 43}$ $\quad \cancel{25 =}\ 9 - \frac{3}{a}$

$\cancel{a = \frac{27}{43}}$ $\quad \varepsilon + \delta + \varphi = 25 = 9 - \frac{3}{a}$

$\cancel{a = \frac{3}{16}}$ $\quad$ so $a = -\frac{3}{16}$

so y intercept $= f(0) = -24a$
$= \frac{9}{2}$

~~this is so skibidi~~

QUESTION THREE

(a) Prove that $\dfrac{1}{1+\sin x} = \sec x(\sec x - \tan x)$.

Hence, or otherwise, find the exact value of:

$$\int_0^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,\mathrm{d}x$$

$(1+\sin x)\sec x(\sec x - \tan x)$

$= \sec x\left(\sec x - \tan x + \frac{\sin x}{\cos x} - \frac{\sin^2 x}{\cos x}\right)$

~~$= \ldots(1 - \tan^2 x)$~~

$= \sec^2 x\left(1 - \frac{\sin^2 x}{\tan^2 x}\right)$

$= \sec^2 x(\cos^2 x) = 1$

so since $(1+\sin x)\sec x(\sec x - \tan x) = 1$

identity true

see back page

(b) (i) It can be shown that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

Find a similar expression for $\tan\left(\frac{3\pi}{8}\right)$.

Show all working.

$\tan\left(\frac{3}{4}\pi\right) = \tan(135°)$ is well known to be (-1)

but also by $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

let $\tan\left(\frac{3\pi}{8}\right)$ be x

$-1 = \frac{2x}{1-x^2}$

$x^2 - 1 = 2x,\ x^2 - 2x - 1 = 0$

 look at back

(ii) Hence, or otherwise, find all real solutions of the following system of equations in exact form:

$$\tan\left(\frac{x}{2}\right)\tan y = \sqrt{2} - 1$$

$$\tan\left(\frac{x}{2} + y\right) = 1 + \sqrt{2}$$

Ohhh isee...

$\tan\left(\frac{x}{2} + y\right) = 1 + \sqrt{2}$

$= \dfrac{\tan\frac{x}{2} + \tan y}{1 - \tan\frac{x}{2}\tan y}$

$= \dfrac{\tan\frac{x}{2} + \tan y}{1 + 1 - \sqrt{2}}$

so $\tan\frac{x}{2} + \tan y = (2-\sqrt{2})(1+\sqrt{2})$

$\tan\frac{x}{2} = \dfrac{\sqrt{2}-1}{\tan y}$ $= \sqrt{2}$

$\dfrac{\sqrt{2}-1}{\tan y} + \tan y = \sqrt{2}$

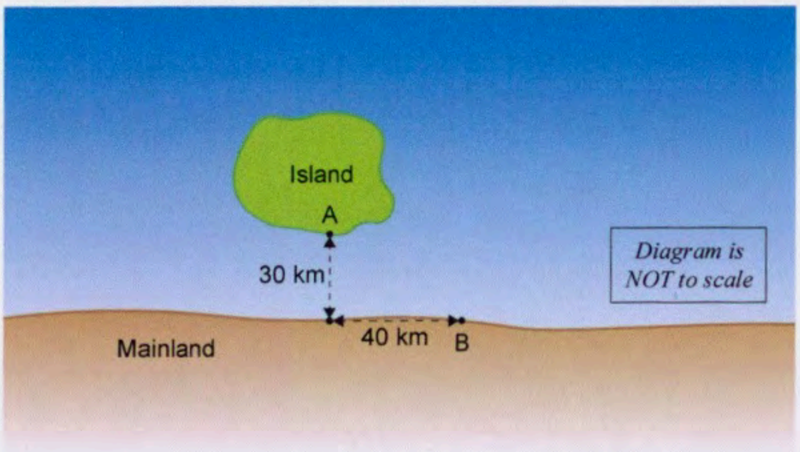
$\tan y^2 - \sqrt{2}\tan y + \sqrt{2} - 1 = 0$

$(\tan y - 1)(\tan y - (1-\sqrt{2})) = 0$

$\boxed{\tan y = 1,\ \tan\frac{x}{2} = \sqrt{2}-1}$, or $\boxed{\tan y = -1 + \sqrt{2},\ \tan\frac{x}{2} = 1}$

so $y = \frac{\pi}{4}$ and $x = \frac{\pi}{4}$ or $y = \frac{\pi}{8}$ and $x = \frac{\pi}{2}$

(c) A local council wishes to supply internet to an island 30 km from the mainland. One option involves running a cable between point A on the island and point B on the mainland, 40 km further down the coast, as shown below.

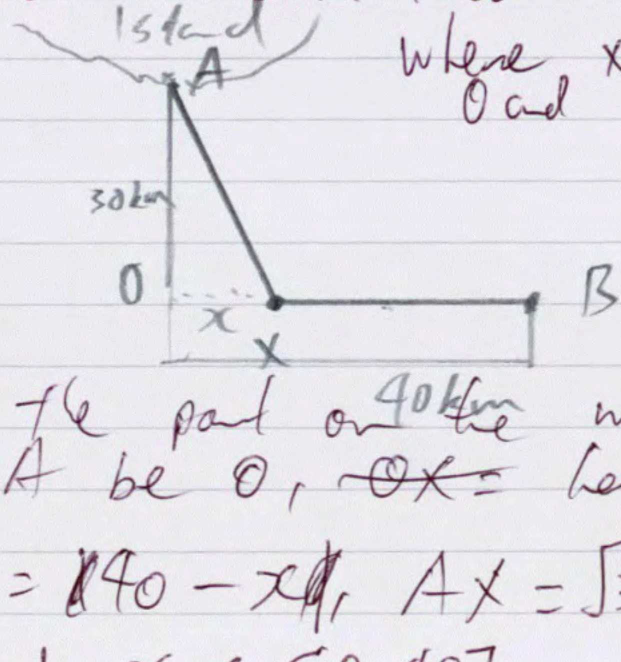


The cost to run a submarine cable under the water is $\$W$ per km, and the cost to run an underground cable on the mainland is $\$L$ per km, where $W > L$.

Use calculus to determine how the cable should be installed to minimise the overall cost of this option for two distinct scenarios:

- $W = 1.5L$
- $W = 1.2L$

the optimal solution (by triangle inequality) should involve a straight line to point X on the coast then under to B



where X is between O and B

let the point on the mainland closest to A be O, ~~OX =~~ let $\overline{OX}$ be x

$\overline{BX} = 40 - x$, $AX = \sqrt{30^2 + x^2}$

and $x \in [0, 40]$

want to minimise cost

~~tt~~ let cost be $C(x)$ for any given x

$C(x) = W\sqrt{\cancel{30}900+x^2} + \cancel{4}L(40-x)\cancel{4}$

$C'(x) = W\left(\frac{\cancel{1}2x}{2\sqrt{900+x^2}}\right) \cancel{+} - L\,\cancel{(sgn(40-x))}$

$= \frac{\cancel{x}Wx}{\sqrt{900+x^2}} - L\,\cancel{(sgn(40-x))}$

$= 0$

case 1, $W = 1.5L$

$\frac{3L\,x}{2\sqrt{900+x^2}} = L\,\cancel{(sgn(40-x))}$

$\frac{3}{2}x = \sqrt{900+x^2}$

$\frac{9}{4}x^2 = 900+x^2$

$\frac{5}{4}x^2 = 900$

$x^2 = 720$

$x = 12\sqrt{5} \approx 26.83$

so approx 26.8 km from O

$13.\cancel{2}\cancel{8}7$ km from B

this is a minimum

as $C(26.83) = 73.541L$ ~~$= 14.17L$~~

and $C(26.00)$ ~~$= 14.98L$~~ $73.548L$

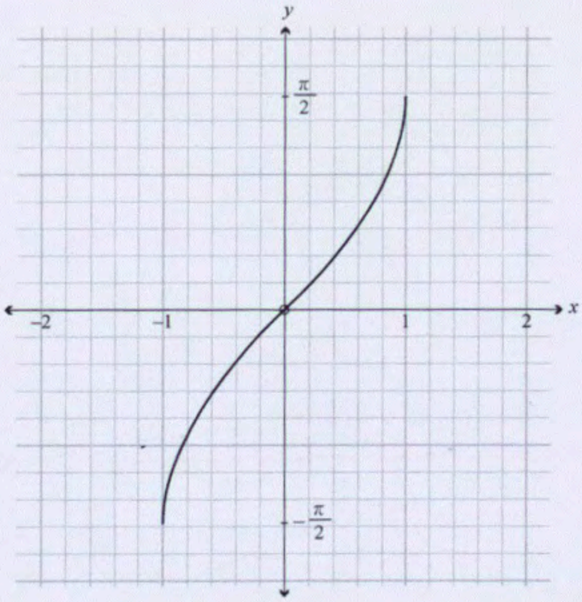
Case 2, $W = 1.2L$

~~$\frac{6Lx}{5\sqrt{900+x^2}} = L \Rightarrow \frac{36}{25}x^2 = 900+x^2, x =$~~

look at back

QUESTION FOUR

(a) The graph below shows $y = \sin^{-1}(x)$, the inverse sine function, on its domain $-1 \leq x \leq 1$.



(i) Evaluate the definite integral:

$\int_0^1 \sin^{-1}(x)\,dx$

$= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin y \, dy$ ~~(tops area~~ (area of rectangle)

$= \frac{\pi}{2} - \left[-\cos y\right]_0^{\frac{\pi}{2}}$

$= \frac{\pi}{2} - [0 + 1]$

$= \frac{\pi}{2} - 1$

yay!

(ii) If $x = \sin y$, show that:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Hence, or otherwise, find the minimum value of $g(x) = \sin^{-1}(2x) + \sin^{-1}\left(\frac{\pi}{4} - 2x\right)$.

You may assume that your answer is indeed a minimum.

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

$x \in D$

$$\cancel{g'(x)} = \text{so } \frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$g'(x) = \frac{2}{\sqrt{1-4x^2}} + \frac{-2}{\sqrt{1-(\frac{\pi}{4}-2x)^2}} = 0$$

$$\text{so } \frac{1}{\sqrt{1-4x^2}} = \frac{1}{\sqrt{1-(\frac{\pi}{4}-2x)^2}}$$

$$\Rightarrow 1-4x^2 = 1-\left(\frac{\pi}{4}-2x\right)^2$$

$$4x^2 = \left(\frac{\pi}{4}-2x\right)^2$$

$$2x = \pm\left(\frac{\pi}{4}-2x\right)$$

Case 1, ~~+~~ +

$$2x = \frac{\pi}{4} - 2x, \quad x = \frac{\pi}{16}$$

Case 2 see back

(b) Consider the function $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot x^r$, where r is a positive, real constant, and $x > 0$.

Find the value of r for which $f(x)$ is a solution of the differential equation $f'(x) = f^{-1}(x)$.

Note: if f and f^{-1} are inverse functions, it can be shown that $f\left(f^{-1}(x)\right) = x$ for all x in the domain of f^{-1}.

$f'(x) = f^{-1}(x)$

$f'(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot r \cdot x^{r-1} = r^{\left(\frac{-r}{r+1}+1\right)} \cdot x^{r-1}$

$\cancel{f^{-1}(x)}$

$f(f^{-1}(x)) = x \Rightarrow x = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot \left(f^{-1}(x)\right)^r$

$x \cdot (r^r)^{\frac{1}{r+1}} = f^{-1}(x)^r$

$(x)^{\frac{1}{r}} \cdot (r)^{\frac{1}{r+1}} = f^{-1}(x)$

$f'(x) = \cancel{x}\, r^{\frac{-r}{r+1}} \cdot r \cdot x^{r-1}$

$= r^{\left(\frac{-r}{r+1}+1\right)} \cdot x^{r-1} = x^{\frac{1}{r}} \cdot r^{\left(\frac{1}{r+1}\right)}$

~~what the sigma~~

Ok maybe i see it

for to be true $\forall x \in \mathbb{R}^+$

$\cancel{r =}$

$r-1 = \frac{1}{r} = \cancel{\frac{\sqrt{5}+1}{2}}$

$\cancel{r = \frac{1+\sqrt{5}}{2}}$

~~and cont'u n-~~

~~And $\frac{-r}{r+1}+1=\frac{1}{r+1}$~~

~~$\Rightarrow -r+r+1=1$~~

~~$2r+1=1$~~

~~$r=0 \Rightarrow$~~

no! what?

well $f(f'(x))=x$

$=r^{\frac{-r}{r+1}}\cdot\left(r^{\frac{-r}{r+1}+1}\cdot x^{r-1}\right)^r=x$

$=r^{\left(\frac{-r}{r+1}+\frac{-r^2}{r+1}+r\right)}\cdot x^{r(r-1)}=x$

~~so $r^{\left(\frac{r}{r+1}+\right.}$~~ $\Rightarrow=\frac{1}{r+1}(-r-r^2+r^2+1)=0$

$r^{\left(\frac{-r}{r+1}+\frac{-r^2}{r+1}+r\right)}\cdot x^{r^2-r-1}=1$

so $x^{r^2-r-1}=1$

~~either $r=0$ which it isn't~~

~~or $x^{r^2-r-1}=0$~~ so $r^2-r-1=0$

since this is true $\forall x\in\mathbb{R}^+$

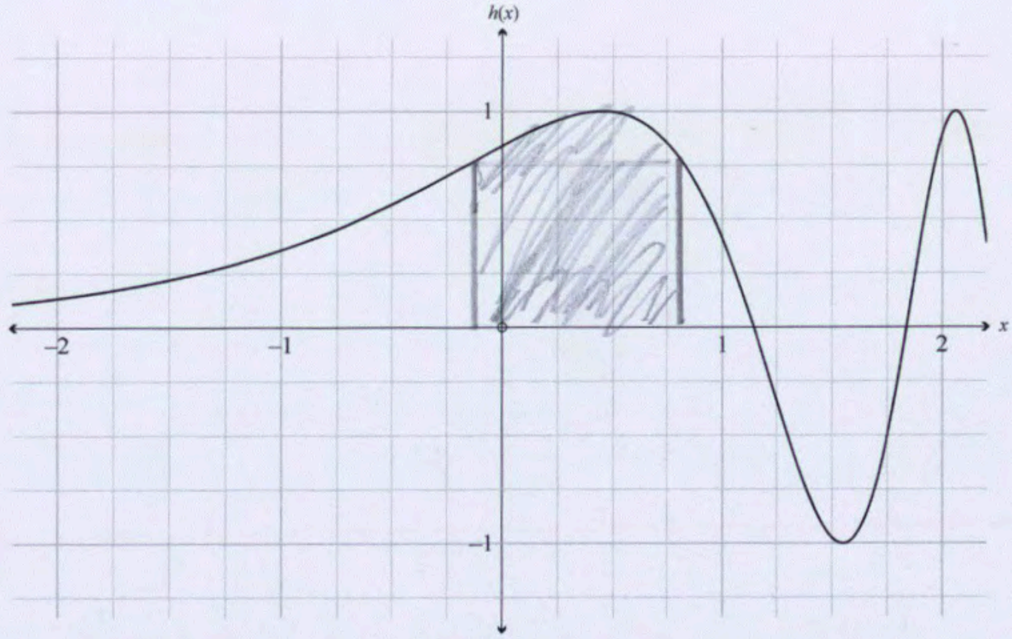
and left side is const, right side (x^{r^2-r-1}) is const

$r^2-r-1=0 \Rightarrow r=\frac{1\pm\sqrt{5}}{2}$

which makes $\frac{-r}{r+1}+\frac{-r^2}{r+1}+r=0$ as well so ϕ works

wait that's the golden ratio!

(c) The graph below shows part of the graph of $h(x) = \sin(e^x)$.



(i) Find the value of $\lim_{x \to -\infty} h(x)$ and $\lim_{x \to \infty} h(x)$, or state clearly if any do not exist.

Support your answer with mathematical reasoning.

$e^{-\infty} = 0$ so ~~the~~ $\lim_{x \to -\infty} \sin(e^x) = 0$

On the right side, $\forall L$, there is always an $x_1 > L$ where $\sin(e^{x_1}) = 1$ and an $x_2 > L$ where $\sin(e^{x_2}) = -1$ so this limit does not converge as ε in the ε-~~δ~~ definition ≥ 1

(ii) Find the exact value of a that generates the maximum value of the definite integral:

$I = \int_a^{a+1} \sin(e^x)\,dx$ Ok what is this

You may use the graph above to justify your answer.

let ~~I(x) be~~ $F(x)$ be $\int_0^x \sin(e^t)\,dt$

$I(x)$ be $F(x+1) - F(x)$

$I'(x) = F'(x+1) - F'(x) = 0$

So $\sin(e^{x+1}) - \sin(e^{x}) = 0$

$\sin e^{x+1} = \sin e^{x} \Rightarrow e^{x+1} = n\pi + (-1)^n e^{x}$

toward the right, area above the and green below the are approximately equal. So the global maximum should be near the origin.

If between e^{x+1} and e^x, there is a full oscillation of the sin function, ~~an~~ ~~$I \approx 0$~~ $I \approx 0$

If more than 1, any peaks approximately cancel any troughs leaving at most 1 small extra peak.

In state maximum, e^x to e^{x+1} should cover ~~half~~ ~~factor~~ a period of the wave that is to say less than

$e^{x+1} = \pi - e^{x}$

$\cancel{e(e^x} \; e^{x}(1+e) = \pi$

$e^{x} = \frac{\pi}{1+e}$, $x = \ln\frac{\pi}{1+e}$

which as seen on the graph is a maximum.
see sketch

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

1d

alternate solution to area of ABCD

$|AOD| = 2 \times 2 \times \frac{1}{2} = 2$

$|ABO| = \frac{1}{2} \times 2 \times 2 \times \sin 60^\circ = \sqrt{3}$

~~let E = AO ∩ DC~~

~~|E DO| =~~

$|OCD| = \frac{1}{2} \times 2 \times \frac{1}{2} \times \sin 150^\circ$

$= -\frac{1}{4}$

So $|ABCD| = |AOD| + |ABO| - |OCD|$

$= \frac{7+4\sqrt{3}}{4}$ or $\frac{7}{4} + \sqrt{3}$

3a $\int_0^{\frac{2}{3}\pi} \frac{1}{1+\sin x} dx$

$= \int_0^{\frac{2\pi}{3}} \sec x(\sec x - \tan x) dx$

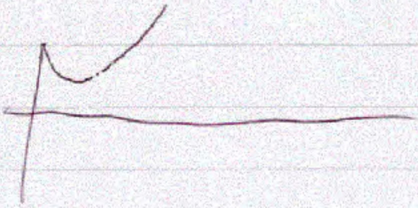
$= \int_0^{\frac{2\pi}{3}} \sec^2 x - \tan x \sec x \, dx$

$= [\tan x - \sec x]_0^{\frac{2\pi}{3}}$ = wait!

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

is there a discontinuity?



newind

$= -\sqrt{3} + 2 - (0 - 1)$

$= 3 - \sqrt{3}$

yay!

3bi $x^2 - 2x - 1 = 0$

$(x - 1)^2 - 1 - 1 = 0$

$(x - 1)^2 = 2$

$x - 1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$

however in this region ~~tan > 0~~
tan is positive $(\theta < \frac{\pi}{2})$

so $x = 1 + \sqrt{2} = \tan\left(\frac{3\pi}{8}\right)$

Extra space if required.
Write the question number(s) if applicable.

QUESTION NUMBER

3C Case 2 $w = \frac{6}{5}L$

$\frac{6}{5}\frac{x}{\sqrt{900+x^2}} = 1$

$\frac{36}{25}x^2 = 900+x^2$

$x \approx 45$

Since > 40

minimum outside domain of our cost function $[0, 40]$
so global minimum in domain is $x = 40$

so straight line from A to B

this is a minimum as $C(40) = 60.96L$
and $C(39) = 60.04L$
and if x were to be right of B
cost is $\frac{6}{5}L\sqrt{900+x^2} + x - 40$
so cost for 1km right is $61.96L$

4aii Case 2, $-$
$2x = -\frac{\pi}{4} + 2x \Rightarrow \frac{\pi}{4} = 0$ contradiction
so $x = \frac{\pi}{16}$

4cii if $n=3$, produces a minimum from x_1 to x_2
if $n=2$, area covers one full period thus small
if $n=4$ covers 2 periods
if $n \geq 5$ etc, area is small as area above to x-axis roughly balances below

93202

Outstanding Scholarship

Subject: Calculus

Standard: 93202

Total score: 31

Q	Score	Marker commentary
1	8	The candidate demonstrated a strong understanding of complex numbers in Q1. They effectively used a diagram to aid in solving a problem involving the Argand plane in Q1(c,) and completed the solution accurately and concisely. They also showed a solid grasp of the Fundamental Theorem of Algebra, applying it with precision.
2	8	The candidate displayed a thorough understanding of functions by skilfully expressing the summation of the function series in Q2(b). They further strengthened their solution by using induction, ensuring the argument was rigorous rather than merely generalised. This approach showcased their solid mathematical knowledge.
3	7	The candidate exhibited strong ability in mathematical modelling and demonstrated excellent understanding of optimisation by comparing turning points with the interval endpoints, particularly in case where the turning point lies outside the problem's domain.
4	8	The candidate demonstrated aptitude in solving Q4(c)(ii) by first finding the general solution of the trigonometric equation. They then carefully analysed the conditions to determine where the maximum area is achieved. Their approach reflected clear logical reasoning and strong analytical skills.